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View of the Derivative Quadrature Formula Norm in Space $W_2^{(4,3)}$

R.G. Rasulov, A. Sattorov, D. T. Mahkamova
Fergana Polytechnic Institute

Abstract: In the paper we consider an extension problem of the Euler-Maclaurin quadrature formula in the space $W_2^{(4,3)}$ by constructing an optimal quadrature formula. The optimal quadrature formula is obtained by minimizing the error of the formula by coefficients at values of the third derivative of a integrand. Using the discrete analogue of the operator $\frac{d^2}{dx^2}-1$ the explicit formulas for the coefficients of the optimal quadrature formula are obtained. Furthermore, it is proved that the obtained quadrature formula is exact for any function of the set $\mathbf{F}=\mathrm{span}\{1,x,x^2,e^x,e^{-x}\}$. Finally, the square of the norm of the error functional for the constructed quadrature formula is calculated. It is shown that the error of the obtained optimal quadrature formula is less than the error of the classical Euler-Maclaurin quadrature formula on the space $L_2^{(4)}$.

Keywords: Optimal quadrature formula, Hilbert space, the error functional, Sobolev method, optimal coefficients, extremal function.

Introduction

It is known that quadrature and cubature formulas can be used for the approximate calculation of definite integrals. In addition, and more importantly, quadrature formulas provide a basic and important tool for the numerical solution of differential and integral equations. The theory of cubature formulas consists mainly of three sections devoted to exact formulas, formulas based on functional-analytic methods, and formulas based on probabilistic methods. In functional-analytic methods, the error between the integral and the corresponding cubature sum is considered as a linear functional on a Banach space and is estimated by the norm of the error functional in the dual Banach space. The norm of the error functional depends on the coefficients and nodes of the formula. The problem of finding the minimum of the error functional norm with respect to coefficients and nodes is called the S.M. Nikolsky problem, and the resulting formula is optimal in the sense of Nikolsky. Minimizing the norm of the error functional with respect to the coefficients at fixed nodes is called Sard's problem. And the resulting formula is called the optimal formula in the sense of Sard. This problem was first studied by A.Sard. Solving these problems in different spaces of differentiable functions, one obtains various types of optimal formulas for numerical integration.

In this paper, we consider the problem of constructing an optimal quadrature formula with Sard derivatives. There are several methods for constructing optimal *Sard* quadrature formulas, such as the spline method, the method of functions, and the Sobolev method. It should be noted that the Sobolev method is based on the use of a discrete analog of a linear differential operator. In different spaces, based on these methods, the Sard problem was studied by many authors. Among these formulas, Euler-Maclaurin-type quadrature formulas are very important for the numerical integration of differentiable functions and are widely used in applications. In various spaces, the optimality of quadrature and cubature formulas of the Euler-Maclaurin type was studied.

One can consider the Euler-Maclaurin quadrature formulas, as well as the extension of the trapezoid rule by including correction terms. It should be noted that in applications and in solving practical problems, numerical integration formulas are of interest for functions with low smoothness.

The present paper is also devoted to extension of the trapezoidal rule and to construction of the optimal quadrature formulas in the sense of Sard.

Statement of the problem

Consider the following quadrature formula

$$\int_0^1 \varphi(x)dx \cong \sum_{\beta=0}^{N} (C_0[\beta]\varphi(h\beta) + C_1[\beta]\varphi'(h\beta) + C_3[\beta]\varphi'''(h\beta)) \qquad (1)$$

where

$$\begin{aligned} C_0[0] &= \frac{h}{2}, \\ C_0[\beta] &= h, \beta = 1,2,...,N-1, \\ C_0[N] &= \frac{h}{2}, \end{aligned} \qquad (2)$$

and

$$\begin{aligned} C_1[0] &= \frac{h^2}{12}, \\ C_1[\beta] &= 0, \beta = 1,2,...,N-1, \\ C_1[N] &= -\frac{h^2}{12}, \end{aligned} \qquad (3)$$

$C_3[\beta]$ are unknown coefficients of the quadrature formula (1), $h = \frac{1}{N}$, N is a natural number.

We suppose that integrands φ belong to $W_2^{(4,3)}$, where by $W_2^{(4,3)}$ is the class of all functions φ defined on $[0,1]$ which posses an absolutely continuous third derivative and whose forth derivative is in $L_2(0,1)$. The class $W_2^{(4,3)}$ under the pseudo-inner product

$$\langle \varphi, \psi \rangle_{W_2^{(4,3)}} = \int_0^1 (\varphi^{(4)}(x) + \varphi^{(3)}(x))(\psi^{(4)}(x) + \psi^{(3)}(x))dx$$

is a Hilbert if we identify functions that differ by a linear combination of $1,\ x,\ x^2$ and e^{-x} (see, for example, [1, 16]). The space $W_2^{(4,3)}$ is equipped by the corresponding norm

$$\|\varphi\|_{W_2^{(4,3)}} = \left[\int_0^1 (\varphi^{(4)}(x) + \varphi^{(3)}(x))^2 dx\right]^{1/2}. \qquad (4)$$

The error of the formula (1) is the difference

$$(\ell,\varphi)=\int_0^1\varphi(x)dx-\sum_{\beta=0}^{N}(C_0[\beta]\varphi(h\beta)+C_1[\beta]\varphi'(h\beta)+C_3[\beta]\varphi'''(h\beta)) \qquad (5)$$

and it defines a functional

$$\ell(x)=\varepsilon_{[0,1]}(x)-\sum_{\beta=0}^{N}(C_0[\beta]\delta(x-h\beta)-C_1[\beta]\delta'(x-h\beta)-C_3[\beta]\delta'''(x-h\beta)), \quad (6)$$

which is called *the error functional* of the quadrature formula (1), where $\varepsilon_{[0,1]}(x)$ is the indicator of the interval $[0,1]$, δ is Dirac's delta-function. The value of the error functional ℓ at a function φ is calculated as $(\ell,\varphi)=\int_{-\infty}^{\infty}\ell(x)\varphi(x)dx$ (see [21]) and ℓ is a linear functional in $W_2^{(4,3)*}$ space, where $W_2^{(4,3)*}$ is the conjugate space to the space $W_2^{(4,3)}$.

Since the error functional (6) is defined on the space $W_2^{(4,3)}$ it is necessary to impose the following conditions for the error functional ℓ

$$(\ell,1):=1-\sum_{\beta=0}^{N}C_0[\beta]=0, \qquad (7)$$

$$(\ell,x):=\frac{1}{2}-\sum_{\beta=0}^{N}C_0[\beta]h\beta-\sum_{\beta=0}^{N}C_1[\beta]=0, \qquad (8)$$

$$(\ell,x^2):=\frac{1}{3}-\sum_{\beta=0}^{N}C_0[\beta](h\beta)^2-2\sum_{\beta=0}^{N}C_1[\beta]h\beta=0, \qquad (9)$$

$$(\ell,e^{-x}):=\int_0^1 e^{-x}dx-\sum_{\beta=0}^{N}(C_0[\beta]e^{-h\beta}-C_1[\beta]e^{-h\beta}-C_3[\beta]e^{-h\beta})=0. \qquad (10)$$

The equations (7)-(10) show that the quadrature formula (1) is exact to the functions $1,x,x^2$ and e^{-x}. One can see that the coefficients $C_0[\beta]$ and $C_1[\beta]$, defined by equalities (2) and (3), satisfy equations (7), (8) and (9). Therefore for unknown coefficients $C_3[\beta],\beta=0,1,...,N$, we have only equation (10).

The absolute value of the error (5) is estimated from above by the norm

$$\|\ell\|_{W_2^{(4,3)*}}=\sup_{\varphi,\|\varphi\|_{W_2^{(4,3)}}\neq 0}\frac{|(\ell,\varphi)|}{\|\varphi\|_{W_2^{(4,3)}}}$$

of the error functional ℓ as follows

$$|(\ell,\varphi)|\le\|\varphi\|_{W_2^{(4,3)}}\cdot\|\ell\|_{W_2^{(4,3)*}}.$$

Furthermore, one can see from (5) that the norm of the error functional (6) depends on coefficients $C_2[\beta]$.

Thus, in order to construct *an optimal quadrature formula* of the form (1) in the sense of Sard we have to solve the following problem.

Problem 1. Find the minimum for the norm of the error functional (6) by coefficients $C_3[\beta]$, i.e.

$$\left\|\ell\right\|_{W_2^{(4,3)*}} = \inf_{C_3[\beta]} \left\|\ell\right\|_{W_2^{(4,3)*}}. \tag{11}$$

The coefficients satisfying equality (11) are called *optimal coefficients* and are denoted as $C_3[\beta]$, $\beta = 0,1,...,N$.

For solving Problem 1, first, we find an expression for the norm of the error functional (6) and next, we calculate it's minimum by coefficients $C_3[\beta]$, $\beta = 0,1,...,N$.

The rest of the paper is organized as follows:

The norm of the error functional ℓ

To get a representation of the norm of the error functional (6) in the space $W_2^{(4,3)}(0,1)$ we use *the extremal function* for this functional (6) which satisfies the following equality:

$$(\ell,\psi_\ell) = \|\ell\|_{W_2^{(4,3)*}} \cdot \|\psi_\ell\|_{W_2^{(4,3)}}.$$

In [2], for the extremal function ψ_ℓ of the error functional ℓ in the space $W_2^{(4,3)}$ we get the following formula

$$\psi_\ell(x) = \ell(x) * G_4(x) + P_2(x) + de^{-x}, \tag{12}$$

where

$$G_4(x) = \frac{\operatorname{sgn} x}{2}\left(\frac{e^x - e^{-x}}{2} - \frac{x^5}{120} - \frac{x^3}{6} - x\right), \tag{13}$$

$P_2(x) = p_2x^2 + p_1(x) + p_0$ is a linear polynomial and d is a real number.

Furthermore, from the results of [16] we have $\|\ell\|_{W_2^{(4,3)*}} = \|\psi_\ell\|_{W_2^{(4,3)}}$ and

$$\|\ell\|^2_{W_2^{(4,3)*}} = (\ell,\psi_\ell). \tag{14}$$

Hence, taking into account equalities (6) and (12) we come to the following expression for the norm of ℓ:

$$\|\ell\|^2 = (\ell,\psi_\ell) = -\sum_{\beta=0}^{N}\sum_{\gamma=0}^{N} C_3[\beta]C_3[\gamma]G_1(h\beta - h\gamma) +$$

$$+2\sum_{\beta=0}^{N} C_3[\beta]\left(\int_0^1 G_3'(x-h\beta)dx - \sum_{\gamma=0}^{N} C_1[\gamma]G_2(h\beta-h\gamma) + \sum_{\gamma=0}^{N} C_0[\gamma]G_3'(h\beta-h\gamma)\right) +$$

$$+\sum_{\beta=0}^{N}\sum_{\gamma=0}^{N} C_0[\beta]C_0[\gamma]G_4(h\beta-h\gamma) - 2\sum_{\beta=0}^{N}\sum_{\gamma=0}^{N} C_0[\beta]C_1[\gamma]G_4'(h\beta-h\gamma) -$$

$$-\sum_{\beta=0}^{N}\sum_{\gamma=0}^{N} C_1[\beta]C_1[\gamma]G_3(h\beta-h\gamma) -$$

$$-2\sum_{\beta=0}^{N} C_0[\beta]\int_0^1 G_4(x-h\beta)dx + 2\sum_{\beta=0}^{N} C_1[\beta]\int_0^1 G_4'(x-h\beta)dx + \int_0^1\int_0^1 G_4(x-y)dxdy \quad (15)$$

where $G_4(x)$ is defined by (13),

$$G_4''(x) = G_3(x) = \frac{\mathrm{sgn}x}{2}\left(\frac{e^x - e^{-x}}{2} - \frac{x^3}{6} - x\right),$$

$$G_4^{(4)}(x) = G_2(x) = \frac{\mathrm{sgn}x}{2}\left(\frac{e^x - e^{-x}}{2} - x\right),$$

$$G_4^{(6)}(x) = G_1(x) = \frac{\mathrm{sgn}x}{2}\left(\frac{e^x - e^{-x}}{2}\right). \quad (16)$$

Thus, we have calculated the norm of the error functional (6).

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